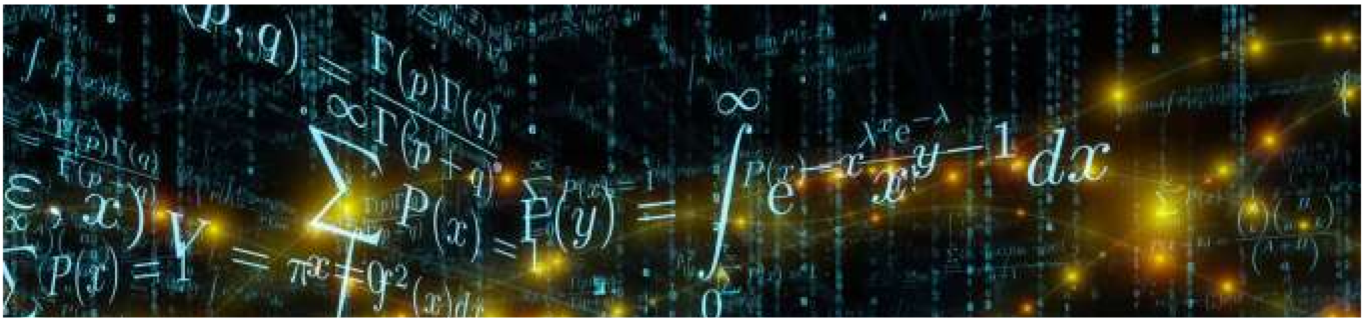
PASSIVITY AND ISS FOR SINGULARLY PERTURBED SYSTEMS WITH EXOGENOUS DISTURBANCE



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Name of the Author:

Dr. Boka Kumsa
Assistant professor, Department of Mathematics, Wollega University, P.O Box 395, Nekemte, ETHIOPIA
Email : kumsab12@yahoo.com

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ABSTRACT

This paper proposes a composite approach between passivity and ISS for singularly perturbed systems (SPSs) with an exogenous input coupled with slow and fast dynamics. The insight links between passivity and ISS are presented explicitly. We first find a linear matrix inequality (LMI) condition under which the SPSs are strictly passive on an exogenous input for all sufficiently small singular perturbation parameters, and then establish a relationship between passivity and input-to-state stability (ISS) for this SPSs with an exogenous input and the problem of designing a robust control law is addressed for such SPSs with a reference input such that the closed loop SPSs are robust passive and ISS on an exogenous input when the singular perturbation parameter is sufficiently small. To achieve this, a matrix inequality-based criterion is proposed and proved. The criterion is then converted into a LMI feasibility issue for the purpose of implementation, in which the control gain matrix is derived by solving such a LMI.

Key Terms:

Exogenous input,
Linear matrix,
Passivity,
Robust,
State stability,
Singular perturbed

I. INTRODUCTION

The fast and slow dynamic phenomena occurring in multiple time-scale system is commonly encountered in real engineering applications. It brings great challenge to control theory since multiple time scales can cause problems of increased order and stiffness of systems. To overcome such problems, singular perturbation method is introduced and has been proven to be a powerful tool in system analysis and synthesis according to (P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, 1986), (D. S. Naidu, 2002) and recently (Lei M. and Chenxiao C. 2016), stability analysis and stabilization synthesis problems of the singularly perturbed switched systems (SPSS) have been investigated first, considering stability of the SPSS with stable subsystems by using an average dwell time approach with a full-order piecewise Lyapunov function. Then, the result is extended to the situation that not all the subsystems are Hurwitz stable. Furthermore, state feedback controllers are designed when all subsystems are stabilizable, and the design method is extended to the situation that not all subsystems are stabilizable. (L. Moreau and D. Aeyels, 2000) considered the stability of singularly perturbed systems with time delays that if the boundary layer system is robustly globally asymptotically stable and the reduced slow system admits a Lyapunov-Razumikhin function, then the singularly perturbed system has certain "practical and semi-global" asymptotic stability property as the parameter of singular perturbation reduces to zero and other many more results have been forwarded.

Motivated by the aforementioned works, this paper focuses on the robust control problem of singularly perturbed systems with exogenous inputs. In this paper, we will study the robust stability for singularly perturbed systems from a passivity point of views. First, using LMI techniques, sufficient conditions for strict passivity of the system without control inputs are given. Meanwhile, the relation between the passivity and input-to-state stability (ISS) is further established. Then, we address the problem of designing a state feedback controller so that the closed-loop system is strictly passive and ISS for sufficiently small perturbation parameters. The contribution of this paper can be summarized as below.

1. A more general class of system is considered, in which exogenous inputs depend on the fast and slow dynamics.
2. The relation between the passivity and ISS is established, it will be especially helpful for us to analyze the stability of systems.
3. A LMI based sufficient condition for passivity and ISS stability of system is provided.

The control gain matrix can be solved efficiently, in which solving much more complex equation is not involved.
The rest of the paper is organized as follows. Section 2 gives the problem formulation. The main results are given in Section 3. Section 4 gives two examples to show the effectiveness of the proposed methods. Finally, the conclusion is drawn in Section 5.

II. PROBLEM FORMULATION

Consider the following nonlinear uncertain SPSs with exogenous disturbances given by

$$\dot{} \qquad {}_{1}+\Delta A_{11}(t))x_1(t)+(A_{12}+\Delta A_{12}(t))x_2(t)+B_{11}w(t)+(B_{12}+\Delta B_{12}(t))u(t), \qquad \text{.........}\ (1)$$

$$\varepsilon\dot{} \qquad {}_{21}+\Delta A_{21}(t))x_1(t)+(A_{22}+\Delta A_{22}(t))x_2(t)+B_{21}w(t)+(B_{22}+\Delta B_{22}(t))u(t), \qquad \text{.........}\ (2)$$

where $x=(x_1^T,x_2^T)^T$ is the system state, $x_1\in R^{n_1}$ and $x_2\in R^{n_2}$ ($n_1+n_2=n$) are the slow state and the fast state, respectively; $u\in R^q$ is the control input; $w\in R^m$ is the bounded disturbance input; $\varepsilon>0$ is a perturbation parameter which is small and positive but may be unknown, representing the response of the fast dynamics. $x_1(t_0)=x_{10}$ and $x_2(t_0)=x_{20}$ are initial conditions A_{ij} and B_{ij} ($i,j=1,2$) are constant matrices with appropriate dimensions; $\Delta A_{ij}(t)$ and $\Delta B_{ij}(t)$ ($i,j=1,2$ are vector-value time- varying uncertainties, which are assumed to be norm bounded, i.e.,

$$\begin{pmatrix}\Delta A_{11}(t) & \Delta A_{12}(t) & \Delta B_{12}(t)\\ \Delta A_{21}(t) & \Delta A_{22}(t) & \Delta B_{21}(t)\end{pmatrix}=\begin{pmatrix}H_1\\ H_2\end{pmatrix}F(t)\begin{pmatrix}E_1 & E_2 & E_3\end{pmatrix}\ , \qquad \text{.........}\ (3)$$

Where, H_1, H_2, E_1 , E_2 and E_3 are constant matrices with appropriate dimensions. $F(t)$ Is an unknown time-varying satisfying

$$F^T(t)F(t)\le I\ ,\ t\in[0,\infty)\ . \qquad \text{.........}\ (4)$$

Define

$$x=\begin{pmatrix}x_1\\ x_2\end{pmatrix},\ E_\varepsilon=\begin{pmatrix}I & O\\ O & \varepsilon I\end{pmatrix},\ A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\ \Delta A(t)=\begin{pmatrix}A_{11}(t) & A_{12}(t)\\ A_{21}(t) & A_{22}(t)\end{pmatrix},$$

$$H=\begin{pmatrix}H_1\\ H_2\end{pmatrix},\ E=\begin{pmatrix}E_1 & E_2\end{pmatrix},\ B_1=\begin{pmatrix}B_{11}\\ B_{21}\end{pmatrix},\ B_2=\begin{pmatrix}B_{12}\\ B_{22}\end{pmatrix},\ \Delta B_2(t)=\begin{pmatrix}B_{12}(t)\\ B_{22}(t)\end{pmatrix}.$$

Then system (1)-(2) can be rewritten in a compact form

$$E_\varepsilon\dot{} \qquad +\Delta A(t))x+B_1w(t)+(B_2+\Delta B_2(t))u(t)\ . \qquad \text{.........}\ (5)$$

It's easy to verify that the uncertain item $\Delta A(t)$ and $\Delta B_2(t)$ satisfy

$$\Delta A\,(t)=HF(t)E\ ,\ \Delta B_2\,(t)=HF(t)E_3\ . \qquad \text{.........}\ (6)$$

We now give some basic materials before continuing our discussion, which will be used in the derivations of the main results.

Definition 2.1:
System (1)-(2) with $u = 0$ is said to be a standard singularly perturbed system, if the algebraic equation

$$0 = (A_{21} + \Delta A_{21}(t))x_1(t) + (A_{22} + \Delta A_{22}(t))x_2(t) + B_{21}w(t) \qquad ……… (7)$$

has a unique isolate root $x_2 = \varphi(t, x_1, w)$ for any given (x_1, w).

From the Definition 2.1, we can see that the existence of the isolate root ensures that the n_1-dimensional reduced model is well defined. Thus it becomes a standard requirement for most singularly perturbed control systems, according to (P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, 1986), (D. S. Naidu, 2002).

Definition 2.2:
*According to (*H. K. Khalil, 2000) Consider the system:

$$w), \qquad ……… (8)$$

Where, state $x(t)$ is in R^n, and control input $w(t)$ in R^m. $f : [0, \infty) \times R^n \times R^m \to R^n$ is continuous and locally Lipchitz in x and w. The input w is a bounded function for all $t \geq 0$. Then the system is said to be input-to-state stable (ISS) if there exist a class KL function β and a class K function γ such that for any initial state $x(t_0)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies:

$$\| x(t) \| \leq \beta(\| x(t_0) \|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \| w(\tau) \|).$$

Remark
The last inequality guarantees that for any bounded input $w(t)$, the state $x(t)$ will be bounded, and as t increases, the state $x(t)$ will be ultimately bounded by a class K function of $\| w \|$. Furthermore, the inequality also shows that if $w(t)$ converges to zero as $t \to \infty$, so does $x(t)$, which will be verified in the next section.

Lemma 2.1:
*According to (*H. K. Khalil, 2000), Let $V : [0, \infty) \times R^n \to R$ be a continuously differentiable function such that

$$\alpha_1(\| x \|) \leq V(t, x) \leq \alpha_2(\| x \|),$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, w) \leq -W(x), \ \| x \| \geq \rho(\| w \|),$$

Where, α_1, α_2 are class K_∞ functions, ρ is a class K function, and $W(x)$ is a continuous positive definite function on R^n. Then, the system (8) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \quad \circ$.

Lemma 2.2:
*According to (*H. K. Khalil, 2000), Let Σ_1 and Σ_2 be real matrices of appropriate dimensions. Then for any matrix F satisfying $F^T F \leq I$ and a scalar $\sigma > 0$,

$$\Sigma_1 F \Sigma_2 + (\Sigma_1 F \Sigma_2)^T \leq \sigma^{-1} \Sigma_1 \Sigma_1^T + \sigma \Sigma_2^T \Sigma_2.$$

III. RESULTS AND DISCUSSION

A. The Existence of an Isolate Root Analysis

In the subsequent two subsections, we will consider system (1)-(2) without control input. The following result presents a sufficient condition in terms of linear matrix inequality to guarantee the existence of an isolate root for the system (1)-(2).

Lemma 3.1:
If there exist a scalar $\sigma > 0$, matrices P_{11}, P_{21} and P_{22} such that the following linear matrix inequality holds

$$\Phi_0 = \begin{pmatrix} A^T P + P^T A + \sigma E^T E & P^T H \\ * & -\sigma I \end{pmatrix} < 0, \qquad ……… (9)$$

Where, $P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix}$. Then the uncertain system (1)-(2) is a standard singularly perturbed system.

Proof:
Let $\varepsilon = 0$, we obtain the reduced-order system from (5)

$$E_0 \dot{} \qquad A(t))x + B_1 w(t),$$

Where, $E_0 = diag\{I_{n_1}, O\}$. Condition (9) implies

$$A^T P + P^T A + \sigma E^T E < 0.$$

Noticing that $\sigma E^T E$ is non-negative, which means that

$$A^T P + P^T A < 0. \qquad ……… (10)$$

Now, making a partition for (10) gives $A_{22}^T P_{22} + P_{22}^T A_{22} < 0$. So, $A_{22}^T P_{22}$ is non-singular, which implies A_{22} is non-singular too.

According to (L. Dai, 1989), it is easy to obtain that the pair (E_0, A) is regular and impulse free, thus there exist matrices $M_1 \in R^{n_1 \times n}$, $M_2 \in R^{n_2 \times n}$, $N_1 \in R^{n \times n_1}$, $N_2 \in R^{n \times n_2}$ such that $M = (M_1^T, M_2^T)^T$ and $N = (N_1, N_2)$ are non-singular upper- and lower triangular matrices, respectively, and the following decomposition holds:

$$ME_0N = diag(I_{n_1}, 0)\,,\ MAN = diag(A_1, I_{n_2})\,,$$

Where, $A_1 \in R^{n_1 \times n_1}$. Noticing that $M_2HH^TM_2^T$ is positive semi definite, thus one has $Q_\xi = (M_2HH^TM_2^T + \xi I)^{-\frac{1}{2}}$ is positive definite for any $\xi > 0$. Let $T_0 = diag(I_{n_1}, Q_\xi)$, $\overline{M} = T_0 M$, $\overline{N} = NT_0^{-1}$, then we have

$$\overline{M}E_0\overline{N} = diag(I_{n_1}, 0)\,,\ \overline{M}A\overline{N} = diag(A_1, I_{n_2})$$

and

$$Q_\xi M_2HH^TM_2^TQ_\xi = (M_2HH^TM_2^T + \xi I)^{-\frac{1}{2}}(M_2HH^TM_2^T)\ (M_2HH^TM_2^T + \xi I)^{-\frac{1}{2}}$$

$$< (M_2HH^TM_2^T + \xi I)^{-\frac{1}{2}}(M_2HH^TM_2^T + \xi I)(M_2HH^TM_2^T + \xi I)^{-\frac{1}{2}} = I\,,$$

Which implies that $\| Q_\xi M_2 H \| < 1$ By the Schur's complement lemma, it is obtained from LMI (9) that

$$A^TP + P^TA + \sigma F^TF + \sigma^{-1}P^THH^TP < 0\,. \qquad \ldots\ldots\ldots\ (11)$$

Pre- and post-multiplying both sides of (11) with $\overline{N}^T$ and $\overline{N}$, respectively, we obtain that

$$(\overline{M}A\overline{N})^T\,\overline{M}^{-T}P\overline{N} + (\overline{M}^{-T}P\overline{N})^T\,\overline{M}A\overline{N} + \sigma\overline{N}^TF^TF\overline{N} + \sigma^{-1}(\overline{M}^{-T}P\overline{N})^T\,\overline{M}HH^T\overline{M}^T\overline{M}^{-T}P\overline{N} < 0. \quad \ldots\ldots\ldots\ (12)$$

Let

$$\overline{M}^{-T}P\overline{N} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix},$$

Then it is easy to know that the structure of $\overline{M}^{-T}$, P and $\overline{N}$ implies that $P_2 = 0$. By decomposing (12) and further calculation, we have that the block matrix at the second block row and the second block column of the left-hand side of (12) is negative definite that is

$$P_4^T + P_4 + \sigma Q_\xi^{-T}N_2^TF^TFN_2Q_\xi^{-1} + \sigma^{-1}P_4^TQ_\xi M_2HH^TM_2^TQ_\xi^TP_4 < 0$$

It implies that there exists a sufficiently small $\xi > 0$ such that

$$P_4^T + P_4 + \sigma Q_\xi^{-T}N_2^TF^TFN_2Q_\xi^{-1} + \sigma^{-1}P_4^TQ_\xi(M_2HH^TM_2^T + \xi I)Q_\xi^TP_4 < 0\,,$$

that is

$$P_4^T + P_4 + \sigma Q_\xi^{-T}N_2^TF^TFN_2Q_\xi^{-1} + \sigma^{-1}P_4^TP_4 < 0\,,$$

This is equivalent,

$$\sigma^{-1}(P_4 + \sigma I)^T(P_4 + \sigma I) - \sigma I + \sigma Q_\xi^{-T}N_2^TF^TFN_2Q_\xi^{-1} < 0\,.$$

Then, it implies that $Q_\xi^{-T}N_2^TF^TFN_2Q_\xi^{-1} < I$. Thus there exists a sufficiently small scalar $\eta > 0$ such that

$$\| FN_2Q_\xi^{-1} \| < \frac{1}{\sqrt{1+\eta}}\,.$$

In order to show the existence of the isolate root, we introduce a change of coordinates

$$\overline{N}^{-1}x = (x_{11}^T \quad x_{12}^T)^T\,, \qquad \ldots\ldots\ldots\ (13)$$

Where $x_{11} \in R^{n_1}$, $x_{12} \in R^{n_2}$. Then the reduced-order system can be rewritten equivalently as follows:

$$[illegible]\ x_{11} + M_1HF(t)E(N_1x_{11} + N_2Q_\xi^{-1}x_{12}) + M_1B_{11}w, \qquad \ldots\ldots\ldots\ (14)$$

$$0 = x_{12} + Q_\xi M_2HF(t)E(N_1x_{11} + N_2Q_\xi^{-1}x_{12}) + Q_\xi M_2B_{21}w. \qquad \ldots\ldots\ldots\ (15)$$

For any given x_{12}, $\overline{x}_{12} \in R^m$, we have

$$\| Q_\xi M_2HF(t)E(N_1x_{11} + N_2Q_\xi^{-1}x_{12}) - Q_\xi M_2HF(t)E(N_1x_{11} + N_2Q_\xi^{-1}\overline{x}_{12}) \|$$

$$= \| Q_\xi M_2HF(t)EN_2Q_\xi^{-1}(x_{12} - \overline{x}_{12}) \| \le \| Q_\xi M_2H \| \| FN_2Q_\xi^{-1}(x_{12} - \overline{x}_{12}) \| \le \| FN_2Q_\xi^{-1} \| \| x_{12} - \overline{x}_{12} \|$$

$$\le (\sqrt{1+\eta})^{-1} \| x_{12} - \overline{x}_{12} \|\,. \qquad \ldots\ldots\ldots\ (16)$$

According to the fixed-point principle, there exists a unique solution $x_{12} = \phi(t, x_{11}, w)$ for any given (x_{11}, w) from (16). Thus, the existence of isolate root $x_2 = \varphi(t, x_1, w)$ is obtained by (13), that is, the system (1)-(2) is in the standard form, which completes the proof.

Remark :

From the proof of Lemma 3.1, it's noticed that the existence of the isolate root for system (1)-(2) is inherited from the reduced-order system, which is a key step for the two-time scale decomposition technique. However, it is shown in (Yang. Y. Robust, 2013) and (H. K. Khalil, 2000) that the corresponding problem has not been investigated. Thus, this can be viewed as an extension according to (Yang. Y. Robust, 2013), and (H. K. Khalil, 2000), furthermore, we can also show that $x_2 = \varphi(t, x_1, w)$ is Lipschitz with respect to (x_1, w), that is, there exist two scalars $\alpha_1 > 0$ and $\alpha_2 > 0$, satisfying the following constraint

$$\| \varphi(t, x_1, w) \| \le \alpha_1 \| x_1 \| + \alpha_2 \| w \| . \qquad \text{……… (17)}$$

And the mathematical derivation is similar to that of (16), thus the detail is omitted here.
Now, according to Lemma 3.1 and the time-scale property of singularly perturbed systems, the slow subsystem of system (1)-(2) can be described by setting $\varepsilon = 0$

$$\dot{x}_s = (A_{11} + \Delta A_{11}(t))x_s + (A_{12} + \Delta A_{12}(t))\overline{x}_2 + B_{11} w_s,\ x_s(t_0) = x_{10}, \qquad \text{……… (18)}$$

$$0 = (A_{21} + \Delta A_{21}(t))x_s + (A_{22} + \Delta A_{22}(t))\overline{x}_2 + B_{21} w_s, \qquad \text{…….. (19)}$$

Where, $\overline{x}_2 = \varphi(t, x_s, w_s)$ here can be viewed as an intermediate variable, which will contribute to the analysis of ISS for (18)-(19)

Define $\overline{x} = (x_s^T \quad \overline{x}_2^T)^T$, then system (18)-(19) can be rewritten in a compact form

$$E_0 \dot{\overline{x}} = (A_{11} + \Delta A_{11}(t))\overline{x} + B_1 w_s . \qquad \text{……… (20)}$$

Moreover, let $x_f = x_2 - \varphi$, $w_f = w - w_s$, the fast subsystem valid in a boundary layer

$$\frac{dx_f}{d\tau} = (A_{22} + \Delta A_{22}(t))x_f + B_{21} w_f,\ x_f(0) = x_{20} - \varphi \qquad \text{……… (21)}$$

Can be obtained by setting $\varepsilon = 0$ and changing time scale

$$\tau = \frac{t - t_0}{\varepsilon}$$

In system, (1)-(2)
Since (20)-(21) is only an approximate model of system (1)-(2), the ISS property of reduced order system does not imply the one of the original system. Thus, attention is now focused on how the ISS property of the original system (1)-(2) can be deduced from the reduced order systems in separate time scales.

B. Input-to-State Stability Analysis

In this subsection, based on the reduced technique, we will give the sufficient condition under which the full-order system is ISS. First, we have following results for the slow subsystem (20)-(21).

Theorem 3.1:

Under the condition of Lemma 3.1, if the matrix $P_{11} > 0$, then the slow subsystem (20)-(21) is made ISS with respect to the disturbance w_s.

Proof:

In order to show that the slow subsystem (20)-(21) is ISS, we choose the Lyapunov function candidate as follows:

$$S_0(x_s) = x_s^T P_{11} x_s . \qquad \text{…….. (22)}$$

Clearly, $S_0(x_s) > 0$ for any $x_s \ne 0$, Notice that

$$S_0(x_s) = x_s^T P_{11} x_s = \overline{x}^T E_0^T P \overline{x} ,$$

Thus the derivative of S_0 along system (22) yields

$$\dot{S}_0(x_s) = ((A_{11} + \Delta A_{11}(t))\overline{x} + B_1 w_s)^T P \overline{x} + \overline{x}^T P^T ((A_{11} + \Delta A_{11}(t))\overline{x} + B_1 w_s) .$$

Furthermore, for any scalar $\sigma > 0$, using the constraint (3) and Lemma 2.2, we have

$$\dot{S}_0(x_s) = (A\overline{x} + HF(t)E\overline{x} + B_w w_s)^T P \overline{x} + \overline{x}^T P^T (A\overline{x} + HF(t)E\overline{x} + B_w w_s)$$
$$\le \overline{x}^T \Upsilon \overline{x} + 2\overline{x}^T P^T B_1 w_s ,$$

Where, $\Upsilon = A^T P + P^T A + \sigma E^T E + \sigma^{-1} P^T H H^T P$

By the Schur's Complement Lemma, we can see that $\Upsilon < 0$ can be guaranteed by (9). Let $a = \lambda_{\min}(-\Upsilon)$, then it follows from (11) that $a > 0$. 'Thus, it yields

$$\dot{S}_0(x_s) \le -a \| x_s \|^2 + 2\overline{x}^T P^T B_w w_s .$$

Noticing the constraint (19), we further obtain

$$\dot{S}_0(x_s) \le -a \| x_s \|^2 + 2 \| P^T B_w \| (\| x_s \| + \| \varphi \|) \| P^T B_w \| \| w_s \| \le -a \| x_s \|^2 + b \| x_s \| \| w_s \| + c \| w_s \|^2$$

$$\le -a(1-\theta)\| x_s \|^2, \quad \forall \| x_s \| \ge \frac{-b+\sqrt{b^2+4ca\theta}}{2a\theta}\| w_s \|,$$

Where $0<\theta<1$, $b = 2(1+\alpha_1)\| P^T B_w \|$ and $c = 2\alpha_2 \| P^T B_w \|$. Hence, the conditions of Lemma 2.1 are satisfied, and we can conclude that there exist a class KL function β and a class K function γ such that for any initial state x_{10}, the solution $x_s(t)$ exists for all $t \ge t_0$ and satisfies

$$\| x_s(t) \| \le \beta(\| x_s(t_0) \|, t) + \gamma(\sup_{t_0 \le \tau \le t} \| w_s(\tau) \|) . \qquad \text{………} \quad (23)$$

That is, the slow subsystem (20)-(21) is ISS with respect to the disturbance w_S, this is complete proof.

For the fast subsystem (21), we have the following result.

Theorem 3.2:

Under the condition of Lemma 3.1, if the matrix $P_{22} > 0$, then the fast subsystem (21) is made ISS with respect to the disturbance w_f

Proof:

In order to show that the fast subsystem (21) is ISS, We make a partition for LMI (9), which is equivalent to

$$\begin{pmatrix} (1,1) & (1,2) & P_{11}^T H_1 & P_{11}^T H_2 \\ * & (2,2) & O & P_{22}^T H_2 \\ * & * & -\sigma I & O \\ * & * & O & -\sigma I \end{pmatrix} < 0,$$

Where,

$(1,1) = A_{11}^T P_{11} + P_{11}^T A_{11} + A_{21}^T P_{21} + P_{21}^T A_{21} + \sigma E_1^T E_1$, $(1,2) = A_{21}^T P_{22} + P_{11}^T A_{12} + P_{21}^T A_{22} + \sigma E_1^T E_2$,

$(2,2) = A_{22}^T P_{22} + P_{22}^T A_{22} + \sigma E_2^T E_2$.

This indicates that

$$\Phi_1 = \begin{pmatrix} A_{22}^T P_{22} + P_{22}^T A_{22} + \sigma E_2^T E_2 & P_{22}^T H_2 \\ * & -\sigma I \end{pmatrix} < 0 . \qquad \text{………} \quad (24)$$

Let $S_2(x_f) = x_f^T P_{22}\, x_f$, calculating the derivative of S_2 along the trajectories of system (21), we have

$$\dot{S}_2(x_f) = (A_{22}x_f + \Delta A_{22}(t)x_f + B_{21}w_f)^T P_{22} x_f + x_f^T P_{22}^T (A_{22}x_f + \Delta A_{22}(t)x_f + B_{21}w_f) + 2x_f^T P_{22}^T B_{21} w_f$$
$$\le x_f^T (A_{22}^T P_{22} + P_{22}^T A_{22} + \sigma E_2^T E_2 + \sigma^{-1} P_{22}^T H_2 H_2^T P_{22}) x_f + 2x_f^T P_{22}^T B_{21} w_f .$$

Then, the rest of the proof can follow along the same lines of the proof of Theorem 3.1. The exact argument is omitted for brevity.

Remark :

According to (Yanbo Gao, Binghua Sun, and Guoping Lu, 2011), ISS property and asymptotic stability of the slow and fast subsystems were presupposed. Based on these and the assumption of the existence of an isolate root, a type of total stability for ISS property of singularly perturbation was established. Compared with (Yanbo Gao, Binghua Sun, and Guoping Lu, 2011), Theorem 3.1 and 3.2 present verifiable sufficient conditions for ISS of the subsystems via LMIs, which can be regarded as an advance, although the considered system here is less general than that in (Yanbo Gao, Binghua Sun, and Guoping Lu, 2011).

Based on Theorems 3.1 and 3.2, we are now in the position to show the main result of this subsection, which is stated in the following theorem.

Theorem 3.3:

If the condition of Theorems 3.1 and 3.2 hold, then there exists an $\varepsilon^* > 0$, such that the following results hold:

1) System (1)-(2) is a standard singularly perturbed systems;
2) System (1)-(2) is made ISS with respect to disturbance w for any given $\varepsilon \in (0, \varepsilon^*]$.

Proof:

1) The proof of Lemma 3.1 has shown that system (1)-(2) is in standard form, which completes the proof of part 1 2) We now show the ISS property of system (1)-(2). Under the condition of Theorems 3.1 and 3.2, it is shown that both P_{11} and P_{22} are positive definite matrices, then there exists a sufficiently small scalar $\varepsilon_1 > 0$ such that $P_{11} - \varepsilon P_{12}^T P_{22}^{-1} P_{21} > 0$ for all $\varepsilon \in (0, \varepsilon_1]$. By the Schur's Complement Lemma, it yields

$$E_\varepsilon^T P_\varepsilon = P_\varepsilon^T E_\varepsilon = \begin{pmatrix} P_{11} & \varepsilon P_{21}^T \\ \varepsilon P_{21} & \varepsilon P_{22} \end{pmatrix} > 0, \ \varepsilon \in (0, \varepsilon_1],$$

Where, $P_\varepsilon = \begin{pmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{pmatrix}$. Define a Lyapunov function candidate for system (1)-(2) as follows

$$S(x) = x^T E_\varepsilon^T P_\varepsilon\, x, \qquad \text{………} \quad (25)$$

Then, for any a constant $\sigma > 0$, the derivative of

[illegible] $$^{T}P_{\varepsilon}^{T}E_{\varepsilon}\dot{x} = ((A+\Delta A(t))x + B_1 w(t))^T P_{\varepsilon} x + x^T P_{\varepsilon}^T ((A+\Delta A(t))x + B_1 w(t))$$
$$\le x^T (A^T P_{\varepsilon} + PP_{\varepsilon}^T A + \sigma E^T E + \sigma^{-1} P_{\varepsilon} P_{\varepsilon}^T HH^T P)x + 2x^T P_{\varepsilon}^T B_1 w.$$

Where

$$\Upsilon_{\varepsilon} = A^T P_{\varepsilon} + PP_{\varepsilon}^T A + \sigma E^T E + \sigma^{-1} P_{\varepsilon} P_{\varepsilon}^T HH^T P.$$

For the above inequality, by the Schur's complement Lemma, we have that $\Upsilon_{\varepsilon} < 0$ is equivalent to

$$\Phi_0 + \varepsilon \Phi < 0,$$

Where

$$\Phi = \begin{pmatrix} A^T P_0 + P_0^T A & P_0^T H \\ * & O \end{pmatrix}, \; P_0 = \begin{pmatrix} O & P_{21}^T \\ O & O \end{pmatrix}. \qquad \text{........} \quad (26)$$

It follows from (9) that there exists a sufficiently small scalar $\varepsilon_2 > 0$ such that $\Phi_0 + \varepsilon \Phi < 0$ for any given $\varepsilon \in (0, \varepsilon_2]$. Thus, $\Upsilon_{\varepsilon} < 0$ for $\varepsilon \in (0, \varepsilon_2]$ is guaranteed. Let $\overline{a} = \lambda_{\min}(-\Upsilon_{\varepsilon})$, then $\overline{a} > 0$ for $\varepsilon \in (0, \varepsilon_2]$. Further, let $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\}$, then we have $E_{\varepsilon}^T P_{\varepsilon} > 0$ and

[illegible] $$\| x \|^2 + 2x^T P_{\varepsilon}^T B_w w$$

Simultaneously for any given $\varepsilon \in (0, \varepsilon^*]$, thus, we obtain

[illegible] $$\| x \|^2 + 2x^T P_{\varepsilon}^T B_1 w \le -\overline{a}(1-\overline{\theta}) \| x \|^2, \; \| x \| \ge \frac{\max\limits_{\varepsilon \in [0, \varepsilon^*]} \| P_{\varepsilon}^T B_1 \|}{\overline{a}\overline{\theta}} \| w \|, \qquad \text{.........} \; (27)$$

Where, $0 < \overline{\theta} < 1$ hence, the conditions of Lemma 2.1 are satisfied, and we conclude that system (1)-(2) is ISS with respect to the disturbance w for $\varepsilon \in (0, \varepsilon^*]$. That is, there exist a class KL function $\overline{\beta}$, class K function $\overline{\gamma}$ such that for any initial state $x(t_0)$, the solution $x(t)$ exists for all $t \ge t_0$ and satisfies:

$$\| x(t) \| \le \overline{\beta}(\| x(t_0) \|, t - t_0) + \overline{\gamma}(\sup_{t_0 \le \tau \le t} \| w(\tau) \|), \; \varepsilon \in (0, \varepsilon^*]. \qquad \text{.........} \quad (28)$$

So this is completes the proof

Remark 3.3:
Theorem 3.3 presents a unified sufficient for the existence of the isolate root and ISS property for system (1)-(2) by the two-scale decomposition technique and compared with the existing results (Yang. Y. Robust, 2013), our sufficient condition is independent of the coordinate transformation, and only dependent on the solution of LMI (11). Thus, it is easier to test numerically. It is worth pointing out that the derived condition here is independent of ε, therefore, the numerically stiff problem can also be avoided.

As a special case, when the disturbance $w(t)$ satisfies the condition $w(t) \to 0$ as $t \to \infty$, we have the following result.

Corollary 3.1:

Under the condition of Theorem 3.3, if the disturbance $w(t) \to 0$ as $t \to \infty$, then system (1)-(2) is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$.

Proof:
In the case of $w(t) \to 0$ as $t \to \infty$. For any given $\rho > 0$, a scalar $\delta = \delta(\varepsilon, \rho) > 0$ can be found such that $\gamma(\delta) < \rho / 2$. Since $\lim\limits_{t \to \infty} w(t) = 0$, there exists a scalar $T_1 > 0$ such that $\| w(t) \| \le \delta$ for all $t \ge T_1$. Take $t_0 \ge T_1$. For $t \ge t_0$, by the inequality (28), we have

$$\| x(t) \| \le \beta(\| x(t_0) \|, t - t_0) + \gamma(\delta) \le \beta(c, t - t_0) + \rho / 2$$

for some $c > 0$. Since $\beta(c, t - t_0) \to 0$ as $t \to \infty$, there exists a scalar $T_2 > 0$ such that $\beta(c, t - t_0) \le \rho / 2$ for all $t \ge T_2$. Thus this together with the last inequality yields

$$\| x(t) \| \le \rho, \; \forall t \ge T = \max\{T_1, T_2\}.$$

The last inequality shows that $\lim\limits_{t \to \infty} x(t) = 0$, that is, the system (1)-(2) is asymptotically stable for any $\varepsilon \in (0, \varepsilon^*]$. So this is completes the proof.

C. The Input-to-State Stability of Closed-loop Systems

From Theorem 3.3, we know that the system (1)-(2) is ISS with a positive definite Lyapunov function requires that system (1)-(2) be stable when the disturbances and inputs do not exist. However, in many cases, it might not be stable. To remove this restriction, we include feedback as a means to achieve ISS. Instead of being stable, the system is assumed to be stabilizable. Therefore we, in this subsection, need to find a state feedback transformation

$$u = K_1 x_1 + K_2 x_2 , \qquad \cdots\cdots\cdots \quad (29)$$

Where $K = (K_1 \quad K_2)$ is a constant matrix, such that the resulting closed-loop system is ISS with respect to the disturbance $w(t)$ when feedback implementation.

Substituting the above control function (29) into (1)-(2), we obtain the closed-loop system as follows:

$$E_\varepsilon \dot{\ } \qquad + B_2 K + \Delta A + \Delta B_2(t)K)x(t) + B_1 w(t) \qquad \cdots\cdots\cdots \quad (30)$$

Applying theorem 3.3 to the closed-loop system (30), we have the following result.

Theorem 3.4:

If there exist a constant $\sigma > 0$, matrix Y and a lower triangular matrix

$$X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}$$

With $0 < X_{11} \in R^{n_1 \times n_1}$ and $X_{22} \in R^{n_2 \times n_2}$, satisfying the following linear matrix inequality

$$\Omega_0 = \begin{pmatrix} AX + X^T A^T + B_2 Y + Y^T B_2^T + \sigma^{-1} H^T H & X^T E^T + Y^T E_3^T \\ EX + E_3 Y & -\sigma^{-1} I \end{pmatrix} < 0 , \qquad \cdots\cdots\cdots \quad (31)$$

then there exist an $\varepsilon^* > 0$ such that the resulting closed-loop system (30) is in the standard form and ISS with respect to w for any $\varepsilon \in (0, \varepsilon^*]$. Moreover, state feedback gain matrix can be chosen as

$$K = YX^{-1} . \qquad \cdots\cdots\cdots \quad (32)$$

Proof:

By substituting (32) into (31) and using the Schur's Complement Lemma, we obtain that inequality (31) is equivalent to

$$\begin{pmatrix} X^T (A + B_2 K)^T + (A + B_u K)X + \sigma X^T (E + E_3 K)^T (E + E_3 K)X & H \\ * & -\sigma I \end{pmatrix} < 0 . \qquad \cdots\cdots\cdots \quad (33)$$

Pre- and post-multiplying inequality (33) by $diag(X^{-T}, I)$ and $diag(X^{-1}, I)$, respectively, let $X = \overline{P}^{-1}$, $Y = K\overline{P}^{-1}$, then (33) is equivalent to

$$\overline{\Omega}_0 = \begin{pmatrix} (A + B_u K)^T \overline{P} + \overline{P}^T (A + B_u K) + \sigma (E + E_3 K)^T (E + E_3 K) & \overline{P}^T H \\ * & -\sigma I \end{pmatrix} < 0 . \qquad \cdots\cdots\cdots \quad (34)$$

We choose the Lyapunov function candidate as follows

$$V(x) = x^T E_\varepsilon^T \overline{P}_\varepsilon x ,$$

Where

$$\overline{P}_\varepsilon = \overline{P} + \varepsilon \overline{P}_0 , \ \overline{P} = \begin{pmatrix} \overline{P}_{11} & O \\ \overline{P}_{21} & \overline{P}_{22} \end{pmatrix} , \ \overline{P}_0 = \begin{pmatrix} O & \overline{P}_{21}^T \\ O & O \end{pmatrix} .$$

Then, for any a constant $\sigma > 0$, the derivative of

$$\begin{aligned} \dot{V} \qquad {}^{T} \overline{P}_\varepsilon^T E_\varepsilon \dot{\ } &= ((A + B_2 K + \Delta A + \Delta B_2(t)K)x(t) + B_1 w(t))^T \overline{P}_\varepsilon x \\ &+ x^T \overline{P}_\varepsilon^T ((A + B_2 K + \Delta A + \Delta B_2(t)K)x(t) + B_1 w(t)) + 2x^T \overline{P}_\varepsilon^T B_1 w \\ &\le x^T \overline{\Upsilon}_\varepsilon x + 2x^T \overline{P}_\varepsilon^T B_1 w . \end{aligned}$$

Where, $\overline{\Upsilon}_\varepsilon = (A + B_2 K)^T \overline{P}_\varepsilon + \overline{P}_\varepsilon^T (A + B_2 K) + \sigma (E + E_3 K)^T (E + E_3 K) + \sigma^{-1} \overline{P}_\varepsilon^T H H^T \overline{P}_\varepsilon$

For the above inequality, by the Schur's complement Lemma, we have that $\overline{\Upsilon}_\varepsilon < 0$ is equivalent to

$$\overline{\Omega}_0 + \varepsilon \overline{\Omega} < 0 ,$$

Where

$$\overline{\Omega} = \begin{pmatrix} (A + B_2 K)^T \overline{P}_0 + \overline{P}_0^T (A + B_2 K)^T & \overline{P}_0^T H \\ * & O \end{pmatrix} , \ \overline{P}_0 = \begin{pmatrix} O & \overline{P}_{21}^T \\ O & O \end{pmatrix} . \qquad \cdots\cdots\cdots \quad (35)$$

The sequent proof is similar to that of Theorem 3.3, thus there exists a scalar $\varepsilon^* > 0$ such that the closed-loop system (30) is in the standard form and ISS with respect to $w(t)$ for any $\varepsilon \in (0, \varepsilon^*]$. This completes the proof. ■

Similar to Corollary 3.2, in the case of $w(t) = 0$, the closed-loop system (30) reduces to the following form in (H. K. Khalil, 2000).

$$E_{\varepsilon}\dot{\ } \text{[illegible]} + B_2K + \Delta A + \Delta B_2(t)K)x(t) \,. \qquad \ldots\ldots\ldots \qquad (36)$$

The following result can be obtained immediately from theorem 3.4.

Corollary 3.3:

Assume that the conditions of Theorem 3.4 hold, then the resulting closed-loop system (36) is in the standard form and asymptotically stable for any $\varepsilon \in (0, \varepsilon^*]$

IV. CONCLUSION

This paper has investigated the robust passive control for singularly perturbed systems. A proper state feedback control law in terms of LMIs techniques has been presented under which the closed-loop system is strictly passive. Moreover, the relation between the passivity and ISS stability has been established. It's worth mentioning that such a relation in the context of singularly perturbed systems has seldom been reported in the literature.

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