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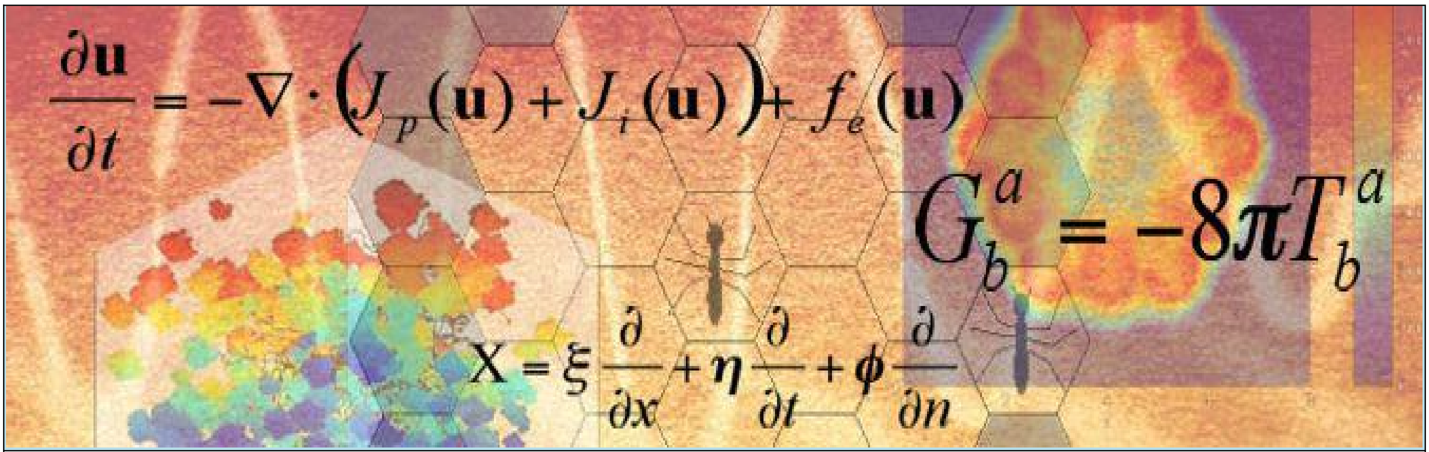

Application to the Fuzzy Fractional Diffusion Equation by using Fuzzy Fractional Variational Homotopy Perturbation Iteration Method

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ABSTRACT

In this article, we present the Fuzzy Fractional Variational Homotopy Perturbation Iteration Method (F-FVHPIM) is applied to solve the fuzzy fractional diffusion equation. The new method is investigated based on the strongly gH-differentiable and Riemann-Liouville gH-differentiability. Using this method, a fast affinity sequence can be obtained that tends to accurately solve the equation. The results show that FFVHPIM is a very effective, convenient, and accurate mathematical tool for solving the fuzzy fractional diffusion equation. Finally, three examples are used to illustrate the main results of this article.

KEYWORDS: Fuzzy numbers; Fuzzy-valued function; Riemann-Liouville gH-differentiability; FFVHPIM; Fuzzy fractional diffusion equation; Mittag-Leffler function

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I. INTRODUCTION

The fractional differential equations are grown increasingly and are considerable in mathematical modeling of physical and engineering topics; for example, see [1], [13], [14], [15], [19]. In related topics, several important contributions such as works and books have been published. Zimmermann [34] presented the fuzzy set theory and its applications. Kilbas et.al [23] introduced the theory and applications of FDEs. Furthermore, several research works have been published to discuss solutions of FDEs; some of them, see [2], [11], [3], [24], [18], [26], [7], [28], [29].

Already many authors have contributed their papers using VIM and HPM for solved some field in a fuzzy environment. Allahviranloo et al. [6] solved nonlinear fuzzy differential equations by using fuzzy VIM. Jameel [21]used VIM for solving fuzzy Duffing's Equation. Amiri [8]proposed VIM for solving a fuzzy generalized Pantograph equation. Tapaswini and Chakraverty [30]presented the numerical solution of n-th order fuzzy linear differential equations by HPM. Narayanamoorthy and Sathiyapriya [25]used the HPM: a versatile tool to evaluate linear and nonlinear fuzzy Volterra integral equations of the second kind. Osman et al. [27]considered the fuzzy ADM and VIM for solving fuzzy heat-like and wave-like equations with variable coefficients defined in the sense of gH-differentiability.

This paper is structured as follows. In Section 2, we call some definitions on fuzzy numbers, fuzzy valued-functions, and Riemann-Liouville gH-differentiability. In Section 3, we present the fuzzy fractional diffusion equation by FFVHPIM. In Section 4, we provide three examples to show the efficiency and simplicity of the way they were developed and derived. Finally, the conclusion is given in section 5.

II. BASIC CONCEPTS

We denote the set of all fuzzy numbers on R by E^1 . A fuzzy number traditionally, is defined [4], [22], [31] by membership function as:

Definition 2.1. A fuzzy number is a function such as $\tilde{u} : R \to E^1$ with the following properties:

- $\tilde{u}$ is normal, i.e. there exists $x_0 \in R$ with $\tilde{u}(x_0) = 1$;
- $\tilde{u}$ is a convex fuzzy set (i.e. $\tilde{u}(\alpha x + (1 - \alpha)y)) \geq \min\{\tilde{u}(x), \tilde{u}(y)\}, \forall \alpha \in [0, 1], x, y \in R$;
- $\tilde{u}$ is semi-continuous on R;
- supp $u = \{x \in R \mid \tilde{u}(x) > 0\}$ is the support of the $\tilde{u}$, and its closure cl (supp $\tilde{u}$) is compact.

Definition 2.2. [31], [5]. A fuzzy number $\tilde{u}$ in parametric forms is a pair $[\underline{u}_r, \bar{u}_r]$ of functions $\underline{u}_r$, $\bar{u}_r$ and $0 \leq r \leq 1$, which satisfy the following properties:

- $\underline{u}_r$ is a bounded non-decreasing left continuous function in (0,1], and right continuous at 0;
- $\bar{u}_r$ is a bounded non-increasing left continuous function in (0,1], and right continuous at 0;
- $\underline{u}_r \leq \bar{u}_r$, $0 \leq r \leq 1$.

Definition 2.3. [20], [34]. Let $\tilde{u}$ be a fuzzy number and $r \in [0, 1]$. The r-level or r-cut set $[\tilde{u}]_r$ is defined as follows:

- $[\tilde{u}]r = \{x \in R \mid \tilde{u}(x) \geq r\}, \quad 0 < r \leq 1$,
- $[\tilde{u}]_0 = cl\{x \in R \mid \tilde{u}(x) > 0\}$.

We denote the r-cut form of a fuzzy number $\tilde{u}$ as $[\tilde{u}]_r = [\underline{u}_r, \bar{u}_r]$. If $\tilde{u}, \bar{v}\ \} \in E^1$ and $\lambda \in R$, then $\tilde{u} + \bar{v}$ and $\lambda\tilde{u}$ are defined by

- $[\tilde{u} \oplus \tilde{v}]_r = [\tilde{u}]_r \oplus [\tilde{v}]_r = [\underline{u}_r + \underline{v}_r, \overline{u}_r + \overline{v}_r]$,
- $[\lambda \odot \tilde{u}]_r = \lambda \odot [\tilde{u}]_r = [\min(\lambda\underline{u}_r, \lambda\overline{u}_r), \max(\lambda\underline{u}_r, \lambda\overline{u}_r)]$

For all $0 \leq r \leq 1$; see [17, 34].

Definition 2.4. [31]. For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1, \tilde{u} = [\underline{u}_r, \overline{u}_r], \tilde{v} = [\underline{v}_r, \overline{v}_r]$, the quantity $D(\tilde{u}, \tilde{v})$= $\sup_{r \in [0,1]} \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}$ is the distance between $\tilde{u}$ and $\bar{v}$ and also the following properties hold:

- (E^1, D) is a complete metric space,
- $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v}, \tilde{w} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e}), \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{0}) \leq D(\tilde{u}, \tilde{0}) + D(\tilde{v}, \tilde{0}), \forall \tilde{u}, \tilde{v} \in E^1$,
- $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v} \in E^1, k \in R$,
- $D(k_1 \odot \tilde{u}, k_2 \odot \tilde{u}) = |k_1 - k_2| D(\tilde{u}, \tilde{0}), \forall \tilde{u} \in E^1, k_1, k_2 \in R$, with $k_1 \cdot k_2 \geq 0$.

Let us recall the definition of the Hukuhara difference (gH-difference) in [32]. Suppose that $\tilde{u}, \tilde{v} \in E^1$. The Hukuhara H-difference has been presented as a set $\tilde{w}$ for which $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \tilde{u} = \tilde{v} \oplus \tilde{w}.$. The H-difference is unique, but it does not always exist (a necessary condition for $\tilde{u} \ominus_{gH} \tilde{v}$ to exist is that $\tilde{u}$ contains a translate $\{c\} \oplus \tilde{v}$ of $\tilde{v}$). A generalization of the Hukuhara difference aims to overcome this situation.

Definition 2.5. [12]The generalized Hukuhara difference between two fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$ is defined as following:

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \begin{cases} \text{(i)}\ \tilde{u} = \tilde{v} \oplus \tilde{w}, \\ or\ \text{(ii)}\ \tilde{v} = \tilde{u} \oplus (-\tilde{w}). \end{cases} \tag{2.1}$$

In terms of the r−levels, we get $[\tilde{u} \ominus_{gH} \tilde{v}] = [\min\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\},\ \max\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}]$ and if the H-difference exists, then $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{u} \ominus_{gH} \tilde{v}$; the conditions for existence of $\tilde{w} = \tilde{u} \ominus_{gH} \tilde{v} \in E^1$ are

$$\text{Case (i)} \quad \begin{cases} \underline{w}_r = \underline{u}_r - \underline{v}_r \text{ and } \overline{w}_r = \overline{u}_r - \overline{v}_r, \forall r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.2}$$

$$\text{Case (ii)} \quad \begin{cases} \underline{w}_r = \overline{u}_r - \overline{v}_r \text{ and } \overline{w}_r = \underline{u}_r - \underline{v}_r, \forall r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.3}$$

It is easy to show that (i) and (ii) are both valid if and only if $\tilde{w}$ is a crisp number. In the case, it is possible that the gH-difference of two fuzzy numbers does not exist. To address this shortcoming, a new difference between fuzzy numbers was introduced in [12]]

Definition 2.6. [16]. Let $\tilde{u}(x,t) : D \to E^1$ and $(x_0, t) \in D$. We say that $\tilde{u}$ is strongly generalized Hukuhara differentiable on (x_0, t) (GH-differentiable for short) if there exists an element $\frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)} \in E^1$ such that

(i) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits (in the metric D)

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h \to 0+} = \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}_{gH}|_{(x_0,t)},$$

or

(ii) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h \to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}_{gH}|_{(x_0,t)},$$

or

(iii) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h \to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}_{gH}|_{(x_0,t)},$$

Or

(iv) for all h > 0 sufficiently small, $\exists \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h \to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}_{gH}|_{(x_0,t)}.$$

Definition 2.7. [9]A fuzzy valued function $\tilde{f}$ of two variables is a rule that assigns to each ordered pair of real numbers, (x, t), in a set D, a unique fuzzy number denoted using $\tilde{f}(x, t)$. The set D is the domain of ˜f and its range is the set of values taken by $\tilde{f}$, i.e., $\{\tilde{f}(x, t) \mid (x, t) \in D\}$.
The parametric representation of the fuzzy valued function $\tilde{f}$: D → E^1 is expressed by $\tilde{f}(x, t; r) = [\underline{f}(x,t;r), \overline{f}(x,t;r)]$, for all $(x, t) \in$ D and $r \in [0, 1]$.

Definition 2.8. [22]A fuzzy-number-valued function $\tilde{f}$: $[a, b] \to E^1$ is said to satisfy the condition (gH) on $[a, b]$ if for any $x_1 < x_2 \in [a, b]$ there exists $\tilde{u} \in E1$ such that $\tilde{f}(x_2) = \tilde{f}(x_1) + \tilde{u}$. We call $\tilde{u}$ the gH-difference of $\tilde{f}(x_2)$ and $\tilde{f}(x_1)$, denoted $\tilde{f}(x_2)$ gH $\tilde{f}(x_1)$. For brevity, we always assume that it satisfies the condition (gH) when we are dealing with the subtraction of fuzzy numbers throughout this work.

Theorem 2.1. [28], [29]. Let $\tilde{f}$ be a fuzzy function on [*a*, ∞) expressed by *r*-level set $[\underline{f}(x;r), \overline{f}(x;r)]$. For any fixed $r \in [0, 1]$, assume $\underline{f}(x;r)$, and $\overline{f}(x;r)$ are Riemann-integrable on [*a*, *b*] for every $b \geq a$, and assume there are two positive functions $\underline{N}(r)$ and $\overline{N}(r)$ such that $\int_a^b |\underline{f}(x;r)| \mathrm{dx} \leq \underline{N}(r)$ and $\int_a^b |\overline{f}(x;r)| \mathrm{dx} \leq \overline{N}(r)$ for every $b \geq a$. Then $\tilde{f}$ (*x*) is fuzzy improper Riemann-integrable on [*a*, ∞) and the fuzzy improper Riemannintegral is a fuzzy number. Furthermore, we get

$$\int_a^\infty \tilde{f}(x;r)\mathrm{dx} = \left(\int_a^\infty \underline{f}(x;r)\mathrm{dx}, \int_a^\infty \overline{f}(x;r)\mathrm{dx}\right) \tag{2.4}$$

We denote C^F [*a*, *b*] as a space of all fuzzy-valued functions which are continuous on [*a*, *b*], and the space of all Lebesgue-integrable fuzzy-valued functions on the bounded interval $[a, b] \subset R$ by L^F [*a*, *b*].

2.1 Fuzzy Riemann-Liouville gH-differentiability

We present some definitions and theorems of fuzzy Riemann-Liouville integrals and derivatives under generalized Hukuhara differentiability. Jest now, we will consider the fuzzy Riemann-Liouville integrals of fuzzy-valued function as the following:

Definition 2.9. [29] Suppose that $\tilde{f}(x) \in C^F[a,b] \cap L^F[a,b]$. The fuzzy Riemann-Liouville integral of fuzzy-valued function $\tilde{f}$ is defined as following

$$J_{a+}^\alpha \tilde{f}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad 0 < \alpha \leq 1, \quad x > a. \tag{2.5}$$

Suppose that us consider the *r*-level set representation of fuzzy-valued function $\tilde{f}$ as $\tilde{f}$ as $f(x;r) = [\underline{f}(x;r), \overline{f}(x;r)]$, for $0 \leq r \leq 1$, then we can indicate the fuzzy Riemann-Liouville integrals of fuzzy-valued function $\tilde{f}$ based on the lower and upper functions as:

Theorem 2.2. [28], [10]. Suppose that $\tilde{f} \in C^F[a,b] \cap L^F[a,b]$ is a fuzzy-valued function. The fuzzy RiemannLiouville integral of a fuzzy-valued function $\tilde{f}$ is defined as:

$$J_{a+}^\alpha \tilde{f}(x;r) = [J_{a+}^\alpha \underline{f}(x;r), J_{a+}^\alpha \overline{f}(x;r)], \quad r \in [0,1], \tag{2.6}$$

Where,

$$J_{a+}^\alpha \underline{f}(x;r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t;r)dt}{(x-t)^{1-\alpha}}, \tag{2.7}$$

$$J_{a+}^\alpha \overline{f}(x;r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t;r)dt}{(x-t)^{1-\alpha}}. \tag{2.8}$$

Jest now, we will consider the fuzzy Riemann-Liouville fractional derivative about order $0 < a \leq 1$, for fuzzyvalued function $\tilde{f}$ (which is a direct extension of strongly generalized gH-differentiability [16] to the fractional literature) as following:

Definition 2.10. [29]. Suppose that $\tilde{f} \in C^F[a,b] \cap L^F[a,b], x_0$ in (a,b) and $\Phi(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^\alpha}$. We say that $\tilde{f}$ is Riemann-Liouville gH-differentiable about order $0 < a \leq 1$ at x_0, if there exists an element $\left({}^{RL}D_{a+}^\alpha \tilde{f}\right)(x_0) \in E^1$, such that for all $r \in [0, 1]$, $h > 0$, sufficiently small.

(1) $${}^{RL}D_{a+}^\alpha \tilde{f}(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0+h) \ominus_{gH} \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus_{gH} \Phi(x_0-h)}{h} \tag{2.9}$$

or

(2) $${}^{RL}D_{a+}^\alpha \tilde{f}(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus_{gH} \Phi(x_0+h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0-h) \ominus_{gH} \Phi(x_0)}{-h} \tag{2.10}$$

or

(3) $$^{RL}D^{\alpha}_{a+}\tilde{f}(x_0) = \lim_{h\to 0^+}\frac{\Phi(x_0+h)\ominus_{gH}\Phi(x_0)}{h} = \lim_{h\to 0^+}\frac{\Phi(x_0-h)\ominus_{gH}\Phi(x_0)}{-h} \quad (2.11)$$

or

(4) $$^{RL}D^{\alpha}_{a+}\tilde{f}(x_0) = \lim_{h\to 0^+}\frac{\Phi(x_0)\ominus_{gH}\Phi(x_0+h)}{-h} = \lim_{h\to 0^+}\frac{\Phi(x_0)\ominus_{gH}\Phi(x_0-h)}{h} \quad (2.12)$$

For sake of simplicity, we say that the fuzzy-valued function $\tilde{f}$ is $^{RL}[(i)-a]$-differentiable if it is differentiable as in the Definition 2.10 case (i), and $\tilde{f}$ is $^{RL}[(2)-a]$-differentiable if it is differentiable as in the Definition 2.10 case (2), etc for other cases.

Theorem 2.4. [11]. Suppose that $\tilde{f}\in C^F[a,b]\cap L^F[a,b], x_0$ in (a,b) and $0<\alpha\leq 1$. Then

(I) Suppose that us consider $\tilde{f}$ is $^{RL}[(1)-a]$-differentiable fuzzy-valued function, then

$$^{RL}D^{\alpha}_{a+}\tilde{f}(x_0;r) = \left[^{RL}D^{\alpha}_{a+}\underline{f}(x_0;r), ^{RL}D^{\alpha}_{a+}\overline{f}(x_0;r)\right], \quad 0\leq r\leq 1.$$

(II) Suppose that us consider $\tilde{f}$ is $^{RL}[(2)-a]$-differentiable fuzzy-valued function, then

$$^{RL}D^{\alpha}_{a+}\tilde{f}(x_0;r) = \left[^{RL}D^{\alpha}_{a+}\overline{f}(x_0;r), ^{RL}D^{\alpha}_{a+}\underline{f}(x_0;r)\right], \quad 0\leq r\leq 1,$$

where

$$^{RL}D^{\alpha}_{a+}\underline{f}(x_0;r) = \left[\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x\frac{\underline{f}(t;r)dt}{(x-t)^{\alpha}}\right]_{x=x_0},$$

and

$$^{RL}D^{\alpha}_{a+}\overline{f}(x_0;r) = \left[\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x\frac{\overline{f}(t;r)dt}{(x-t)^{\alpha}}\right]_{x=x_0}.$$

Theorem 2.5. [29]. Suppose that $\tilde{f}\in C^F[a,b]\cap L^F[a,b]$ a Riemann-Liouville gH-differentiable of order $0 < a \leq 1$ on each point $x\in(a, b)$ in the sense of Definition 2.10 case (3) or case (4). Then $^{RL}D^{\alpha}_{a+}\tilde{f}(x)\in R$ for all $x\in(a, b)$.

III. ANALYSIS OF THE METHOD

In this section, we establish an exact solution for the fuzzy fractional diffusion equation by using FFVHPIM. Consider the following fuzzy time-fractional diffusion equation of the form

$$\frac{\partial^{\alpha}\tilde{u}(X,t)}{\partial t^{\alpha}} = D\Delta\tilde{u}(X,t)\ominus_{gH}\nabla\cdot(F(X)\odot\tilde{u}(X,t)), \quad 0<\alpha\leq 1, \quad D>0, \quad (3.1)$$

subject to the initial condition

$$\tilde{u}(X,t)=\tilde{\varphi}(X), \quad X\in\Psi \quad (3.2)$$

and boundary condition

$$\tilde{u}(X,t)=\tilde{\varphi}(X,t), \quad X\in\partial\Psi, \quad t\geq 0. \quad (3.3)$$

Where $\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\cdot)$ is the modified Riemann-Liouville derivative for $0<\alpha\leq 1$, $\triangle$ is the Laplace operator, ∇ is the Hamilton operator, $\Psi = [0, L_1]\times[0, L_2]\times\cdots\times[0, L_d]$ is the spatial domain of the problem, d is the dimension of the space, $X = (x_1, x_2, x_3, ..., x_d)$, $\partial\Psi$ the boundary of Ψ, $\tilde{u}(X, t)$ denotes the probability density fuzzy-valued function of finding a particle at X in time t, the positive constant D depends on the temperature, the friction coefficient, the universal gas constant and finally on the Avagadro constant, $F(X)$ is the external force. From equation (3.1) can be interpreted as modeling the diffusion of a particle under the action of the external force $F(X)$.

3.1 FFVHPIM

We present some steps illustrate the fuzzy fractional variational homotopy perturbation iteration method as follows.

Step [1]. Let a fuzzy nonlinear equation, say in two independent variables x and t, we define as

$$D_t^{\gamma}\tilde{u}(x,t) = L(\tilde{u}(x,t)) \oplus N(\tilde{u}(x,t)) \oplus \tilde{g}(x,t), \tag{3.4}$$

Here $D_t^{\gamma}(\cdot)$ is the fuzzy Riemann-Liouville gH-differentiability, $\gamma > 0$, L is a linear operator, N is a nonlinear operator, $\tilde{u} = \tilde{u}(x, t)$ is an unknown fuzzy-valued function, and $\tilde{g}(x, t)$ is the fuzzy inhomogeneous term.

Step [2]. Construct the correct functional as follows:

$$\left.\begin{array}{l} \underline{u}_{j+1}(x,t;r) = \underline{u}_j(x,t;r) + J_t^{\gamma}\left[\lambda(\tau)\left(D_\tau^{\gamma}\underline{u}_j(x,\tau;r) - L(\underline{u}_j(x,\tau;r)) - N(\underline{u}_j(x,\tau;r)) - \underline{g}(x,\tau;r)\right)\right] \\ \overline{u}_{j+1}(x,t;r) = \overline{u}_j(x,t;r) + J_t^{\gamma}\left[\lambda(\tau)\left(D_\tau^{\gamma}\overline{u}_j(x,\tau;r) - L(\overline{u}_j(x,\tau;r)) - N(\overline{u}_j(x,\tau;r)) - \overline{g}(x,\tau;r)\right)\right] \end{array}\right\}, \quad j \geq 0 \tag{3.5}$$

where λ is a general Lagrange multiplier, which can be specified optimally via variational theory. The subscript $j \geq 0$ denotes the jth approximation, the fuzzy-valued function $\tilde{u}_j(\tau;r) = [\underline{u}_j(\tau;r), \overline{u}_j(\tau;r)]$ is a restricted variations which means as

$$\delta\underline{u}_j(r) = 0, \quad \delta\overline{u}_j(r) = 0.$$

Step [3]. We construct the following iteration formula, from the fuzzy fractional variational homotopy perturbation iteration method as

$$\sum_{j=0}^{\infty} q^j \underline{u}_j(x,t;r) = \underline{u}_0(x,t;r) + q\left[\sum_{j=0}^{\infty} q^j \underline{u}_j(x,t;r)\right.$$
$$\left. + J_t^{\gamma}\left[\lambda(\tau)\left(\sum_{j=0}^{\infty} q^j D_\tau^{\gamma}\underline{u}_j(x,\tau;r) - \sum_{j=0}^{\infty} q^j L(\underline{u}_j(x,\tau;r)) - \sum_{j=0}^{\infty} q^j N(\underline{u}_j(x,\tau;r)) - \underline{g}(x,\tau;r)\right)\right]\right], \tag{3.6}$$

and

$$\sum_{j=0}^{\infty} q^j \overline{u}_j(x,t;r) = \overline{u}_0(x,t;r) + q\left[\sum_{j=0}^{\infty} q^j \overline{u}_j(x,t;r)\right.$$
$$\left. + J_t^{\gamma}\left[\lambda(\tau)\left(\sum_{j=0}^{\infty} q^j D_\tau^{\gamma}\overline{u}_j(x,\tau;r) - \sum_{j=0}^{\infty} q^j L(\overline{u}_j(x,\tau;r)) - \sum_{j=0}^{\infty} q^j N(\overline{u}_j(x,\tau;r)) - \overline{g}(x,\tau;r)\right)\right]\right], \tag{3.7}$$

which is named as the FFVHPIM. where λ is the general Lagrange multiplier acquired in step [2], $q \in [0, 1]$ is an imbedding parameter, and $\tilde{u}_0(x,t;r) = [\underline{u}_0(x,t;r), \overline{u}_0(x,t;r)]$ is an initial approximation of is equation (3.4).

Step [4]. Compared with the coefficient of the same power of q in both hands of the expression of equations (3.6) and (3.7), we have $u_i (i = 1, 2, 3, \cdots)$. From the fuzzy homotopy perturbation method, for $q \longrightarrow 1$ we obtain

$$\left.\begin{array}{l} \underline{u}(x,t;r) = \underline{u}_0(x,t;r) + \underline{u}_1(x,t;r) + \underline{u}_3(x,t;r) + \cdots \\ \overline{u}(x,t;r) = \overline{u}_0(x,t;r) + \overline{u}_1(x,t;r) + \overline{u}_3(x,t;r) + \cdots \end{array}\right\} \tag{3.8}$$

3.2 Application of the method

We conceder the application of the FFVHPIM. From the fuzzy Riemann-Liouville gH-differentiability and FFVHPIM, construct the following corrected functional for equation (3.1), we define as

$$\begin{aligned}
\underline{u}_{n+1}(X,t;r) &= \underline{u}_n(X,t;r) + J_t^{\alpha}\left[\lambda(\tau)\left(\frac{\partial^{\alpha}\underline{u}_n(X,\tau;r)}{\partial\tau^{\alpha}} - \Delta\underline{u}_n(X,\tau;r) + \nabla\cdot(F(X)\underline{u}_n(X,\tau;r))\right)\right]\\
&= \underline{u}_n(X,t;r) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\lambda(\tau)\left(\frac{\partial^{\alpha}\underline{u}(X,t;r)}{\partial\tau^{\alpha}}\right.\right.\\
&\quad -\Delta\underline{u}_n(X,\tau;r) + \nabla\cdot(F(X)\underline{u}_n(X,\tau;r)))]\,d\tau\\
&= \underline{u}_n(X,t;r) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \left[\lambda(\tau)\left(\frac{\partial^{\alpha}\underline{u}(X,t;r)}{\partial\tau^{\alpha}}\right.\right.\\
&\quad -\Delta\underline{u}_n(X,\tau;r) + \nabla\cdot(F(X)\underline{u}_n(X,\tau;r)))]\,(d\tau)^{\alpha},
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\overline{u}_{n+1}(X,t;r) &= \overline{u}_n(X,t;r) + J_t^{\alpha}\left[\lambda(\tau)\left(\frac{\partial^{\alpha}\overline{u}_n(X,\tau;r)}{\partial\tau^{\alpha}} - \Delta\overline{u}_n(X,\tau;r) + \nabla\cdot(F(X)\overline{u}_n(X,\tau;r))\right)\right]\\
&= \overline{u}_n(X,t;r) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\lambda(\tau)\left(\frac{\partial^{\alpha}\overline{u}(X,t;r)}{\partial\tau^{\alpha}}\right.\right.\\
&\quad -\Delta\overline{u}_n(X,\tau;r) + \nabla\cdot(F(X)\overline{u}_n(X,\tau;r)))]\,d\tau\\
&= \overline{u}_n(X,t;r) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \left[\lambda(\tau)\left(\frac{\partial^{\alpha}\overline{u}(X,t;r)}{\partial\tau^{\alpha}}\right.\right.\\
&\quad -\Delta\overline{u}_n(X,\tau;r) + \nabla\cdot(F(X)\overline{u}_n(X,\tau;r)))]\,(d\tau)^{\alpha},
\end{aligned} \tag{3.10}$$

where $u_0(X,t;r) = [\underline{u}_0(X,t;r), \overline{u}_0(X,t;r)]$ is an initial approximation which must be chosen suitable, $u_n(X, t; r) = [\underline{u}_n(X,t;r), \overline{u}_n(X,t;r)]$ is the confined variation, which means $\delta\underline{u}_n(X,t;r) = 0,\quad \delta\overline{u}_n(X,t;r) = 0$, λ is the general Lagrange multiplier, and its optimal value can be found using variation theory. Conditions can be obtained as

$$\left.\begin{array}{l}\lambda^{\alpha}(\tau) = 0, \quad \tau\in[0,1],\\ 1+\lambda(\tau)\mid_{\tau=t} = 0.\end{array}\right\}$$

Thus, the Lagrange multiplier can be specified as $\lambda = -1$. We can construct the iteration formulation, utilizing equations (3.9) and (3.10), we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} q^j\underline{u}_j(X,t;r) &= \underline{u}_0(X,t;r) + q\left[\sum_{j=0}^{\infty} q^j\underline{u}_j(X,t;r) - \frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\sum_{j=0}^{\infty} q^j\frac{\partial^{\alpha}u_j(X,t;r)}{\partial\tau^{\alpha}} - \sum_{j=0}^{\infty} q^j\triangle\underline{u}_j(X,t;r)\right.\right.\\
&\quad \left.\left.+\sum_{j=0}^{\infty} q^j\nabla\cdot(F(X)\underline{u}_j(X,\tau;r))\right)(dx)^{\alpha}\right],
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\sum_{j=0}^{\infty} q^j\overline{u}_j(X,t;r) &= \overline{u}_0(X,t;r) + q\left[\sum_{j=0}^{\infty} q^j\overline{u}_j(X,t;r) - \frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\sum_{j=0}^{\infty} q^j\frac{\partial^{\alpha}u_j(X,t;r)}{\partial\tau^{\alpha}} - \sum_{j=0}^{\infty} q^j\triangle\overline{u}_j(X,t;r)\right.\right.\\
&\quad \left.\left.+\sum_{j=0}^{\infty} q^j\nabla\cdot(F(X)\overline{u}_j(X,\tau;r))\right)(dx)^{\alpha}\right],
\end{aligned} \tag{3.12}$$

where $\underline{u}_0(X,t)$, $\overline{u}_0(X,t)$ is the initial approximate of equation (3.1), $0 < \alpha \le 1$. Compared with the coefficient of the same power of q in both hands of the equations (3.11) and (3.12), we have $\tilde{u}_i(X,t;r) = [\underline{u}_i X,t;r), \overline{u}_i(X,t;r)]$, where i = 0,1,2,....
Applying to fuzzy homotopy perturbation method, we obtain

$$\left.\begin{array}{l}\underline{u}(X,t;r) = \underline{u}_0(X,t;r) + \underline{u}_1(X,t;r) + \underline{u}_2(X,t;r) + \underline{u}_3(X,t;r) + \cdots\\ \overline{u}(X,t;r) = \overline{u}_0(X,t;r) + \overline{u}_1(X,t;r) + \overline{u}_2(X,t;r) + \overline{u}_3(X,t;r) + \cdots\end{array}\right\} \tag{3.13}$$

IV. EXAMPLES

In this section, we will demonstrate how FFVHPIM can be easily applied to has the exact solutions of the fuzzy fractional diffusion equation as follows.

Example 4.1. Consider the following one-dimensional fuzzy fractional diffusion equation. Suppose that D = 1, $F(x) = -1$, $\Psi = [0, 1]$, then equation (3.1), we get

$$\frac{\partial^\alpha \tilde{u}(x,t)}{\partial t^\alpha} = \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} \oplus \frac{\partial \tilde{u}(x,t)}{\partial x}, \qquad 0 < \alpha \leq 1, \tag{4.1}$$

with the initial condition

$$\tilde{u}(x,0) = [(r-3)^n, (-1-r)^n] \odot \exp(x), \quad x \in [0,1], \quad (n = 1,2,3,\ldots), \tag{4.2}$$

and the boundary conditions

$$\tilde{u}(0,t) = E_\alpha(2t^\alpha), \quad \tilde{u}(1,t) = eE_\alpha(2t^\alpha), \quad t \geq 0. \tag{4.3}$$

The parametric form of (4.1) is

$$\frac{\partial^\alpha \underline{u}(x,t)}{\partial t^\alpha} = \frac{\partial^2 \underline{u}(x,t)}{\partial t^2} + \frac{\partial \underline{u}(x,t)}{\partial t}, \qquad 0 < \alpha \leq 1, \tag{4.4}$$

$$\frac{\partial^\alpha \overline{u}(x,t)}{\partial t^\alpha} = \frac{\partial^2 \overline{u}(x,t)}{\partial t^2} + \frac{\partial \overline{u}(x,t)}{\partial t}, \qquad 0 < \alpha \leq 1, \tag{4.5}$$

for $r \in [0,1]$, where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, and similar to $\overline{u}$.
where

$$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+j\alpha)} (\alpha > 0),$$

is the Mittag-Leffer function in one parameter. According to the iteration formulation for equations (3.11) and (3.12) with the initial condition (4.2), we get

$$\left.\begin{aligned}
q^0 &: \underline{u}_0(x,t;r) = (r-3)^n \exp(x)\\
q^1 &: \underline{u}_1(x,t;r) = \frac{(r-3)^n \exp(x) 2t^\alpha}{\Gamma(1+\alpha)},\\
q^2 &: \underline{u}_2(x,t;r) = \frac{(r-3)^n \exp(x) 2^2 t^{2\alpha}}{\Gamma(1+2\alpha)},\\
q^3 &: \underline{u}_3(x,t;r) = \frac{(r-3)^n \exp(x) 2^3 t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\vdots\\
q^j &: \underline{u}_j(x,t;r) = \frac{(r-3)^n \exp(x) 2^j t^{j\alpha}}{\Gamma(1+j\alpha)},
\end{aligned}\right\} \tag{4.6}$$

and

$$\left.\begin{aligned}
q^0 &: \overline{u}_0(x,t;r) = (-1-r)^n \exp(x)\\
q^1 &: \overline{u}_1(x,t;r) = \frac{(-1-r)^n \exp(x) 2t^\alpha}{\Gamma(1+\alpha)},\\
q^2 &: \overline{u}_2(x,t;r) = \frac{(-1-r)^n \exp(x) 2^2 t^{2\alpha}}{\Gamma(1+2\alpha)},\\
q^3 &: \overline{u}_3(x,t;r) = \frac{(-1-r)^n \exp(x) 2^3 t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\vdots\\
q^j &: \overline{u}_j(x,t;r) = \frac{(-1-r)^n \exp(x) 2^j t^{j\alpha}}{\Gamma(1+j\alpha)},
\end{aligned}\right\} \tag{4.7}$$

Thus, whose limit has a compact form as follows

$$\underline{u}(x,t;r) = \lim_{n\to\infty}\sum_{j=0}^{n}\underline{u}_j(x,t;r) = (r-3)^n\left(\lim_{n\to\infty}\sum_{j=0}^{n}\frac{\exp(x)2^j t^{j\alpha}}{\Gamma(1+j\alpha)}\right), \tag{4.8}$$

$$\overline{u}(x,t;r) = \lim_{n\to\infty}\sum_{j=0}^{n}\overline{u}_j(x,t;r) = (-1-r)^n\left(\lim_{n\to\infty}\sum_{j=0}^{n}\frac{\exp(x)2^j t^{j\alpha}}{\Gamma(1+j\alpha)}\right), \tag{4.9}$$

we can be written as follows:

$$\tilde{u}(x,t;r) = [(r-3)^n, (-1-r)^n] \odot (\exp(x)E_\alpha(2t^\alpha)), \quad 0 \le r \le 1.$$

Example 4.2. Consider the following two-dimensional fuzzy fractional diffusion equation. Suppose that $D = 1$, $F(x,y) = -(x,y)$, $\Psi = [0,1]\times[0,1]$, then equation (3.1), we get

$$\frac{\partial^\alpha \tilde{u}(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 \tilde{u}(x,y,t)}{\partial x^2} \oplus \frac{\partial^2 \tilde{u}(x,y,t)}{\partial y^2} \oplus x\odot\frac{\partial \tilde{u}(x,y,t)}{\partial x} \oplus y\odot\frac{\partial \tilde{u}(x,y,t)}{\partial y} \oplus 2\odot\tilde{u}(x,y,t),$$
$$0<\alpha\le 1, \qquad x\ge 0, \quad y\ge 0. \tag{4.10}$$

with the initial condition

$$\tilde{u}(x,y,0) = [(r-7)^n, (-5-r)^n] \oplus (x+y), \quad x,y\in[0,1], \quad (n=1,2,3,\ldots), \tag{4.11}$$

and the boundary conditions

$$\begin{cases} \tilde{u}(0,y,t) = yE_\alpha(3t^\alpha), & t\ge 0,\\ \tilde{u}(1,y,t) = (1+y)E_\alpha(3t^\alpha), & t\ge 0,\\ \tilde{u}(x,0,t) = xE_\alpha(3t^\alpha), & t\ge 0,\\ \tilde{u}(x,1,t) = (x+1)E_\alpha(3t^\alpha), & t\ge 0, \end{cases} \tag{4.12}$$

The parametric form of (4.10) is

$$\frac{\partial^\alpha \underline{u}(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 \underline{u}(x,y,t)}{\partial x^2} + \frac{\partial^2 \underline{u}(x,y,t)}{\partial y^2} + x\frac{\partial \underline{u}(x,y,t)}{\partial x} + y\frac{\partial \underline{u}(x,y,t)}{\partial y} + 2\underline{u}(x,y,t), \quad 0<\alpha\le 1, \tag{4.13}$$

$$\frac{\partial^\alpha \overline{u}(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 \overline{u}(x,y,t)}{\partial x^2} + \frac{\partial^2 \overline{u}(x,y,t)}{\partial y^2} + x\frac{\partial \overline{u}(x,y,t)}{\partial x} + y\frac{\partial \overline{u}(x,y,t)}{\partial y} + 2\overline{u}(x,y,t), \quad 0<\alpha\le 1, \tag{4.14}$$

for $r \in [0, 1]$, where u stands for $u(x, y, t; r)$, and similar to u. where

$$E_\alpha(t) = \sum_{j=0}^{\infty}\frac{t^j}{\Gamma(1+j\alpha)}(\alpha>0),$$

is the Mittag-Leffer function in one parameter. According to the iteration formula (3.11) and (3.12) with the initial condition (4.11), we get

$$\left.\begin{array}{l} q^0 : \underline{u}_0(x,y,t;r) = (r-7)^n + (x+y)\\ q^1 : \underline{u}_1(x,y,t;r) = (r-7)^n + \dfrac{(x+y)3t^\alpha}{\Gamma(1+\alpha)},\\ q^2 : \underline{u}_2(x,y,t;r) = (r-7)^n + \dfrac{(x+y)3^2t^{2\alpha}}{\Gamma(1+2\alpha)},\\ q^3 : \underline{u}_3(x,y,t;r) = (r-7)^n + \dfrac{(x+y)3^3t^{3\alpha}}{\Gamma(1+3\alpha)},\\ \qquad \vdots\\ q^j : \underline{u}_j(x,y,t;r) = (r-7)^n + \dfrac{(x+y)3^jt^{j\alpha}}{\Gamma(1+j\alpha)}, \end{array}\right\} \tag{4.15}$$

and

$$\left.\begin{aligned}
q^0 &: \overline{u}_0(x,y,t;r) = (-5-r)^n + (x+y)\\
q^1 &: \overline{u}_1(x,y,t;r) = (-5-r)^n + \frac{(x+y)3t^\alpha}{\Gamma(1+\alpha)},\\
q^2 &: \overline{u}_2(x,y,t;r) = (-5-r)^n + \frac{(x+y)3^2t^{2\alpha}}{\Gamma(1+2\alpha)},\\
q^3 &: \overline{u}_3(x,y,t;r) = (-5-r)^n + \frac{(x+y)3^3t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\vdots\\
q^j &: \overline{u}_j(x,y,t;r) = (-5-r)^n + \frac{(x+y)3^jt^{j\alpha}}{\Gamma(1+j\alpha)},
\end{aligned}\right\} \tag{4.16}$$

Thus, we obtain a compact form

$$\underline{u}(x,y,t;r) = \lim_{n\to\infty}\sum_{j=0}^{n}\underline{u}_j(x,y,t;r) = (r-7)^n + \left(\lim_{n\to\infty}\sum_{j=0}^{n}\frac{(x+y)3^jt^{j\alpha}}{\Gamma(1+j\alpha)}\right), \tag{4.17}$$

$$\overline{u}(x,y,t;r) = \lim_{n\to\infty}\sum_{j=0}^{n}\overline{u}_j(x,y,t;r) = (-5-r)^n + \left(\lim_{n\to\infty}\sum_{j=0}^{n}\frac{(x+y)3^jt^{j\alpha}}{\Gamma(1+j\alpha)}\right), \tag{4.18}$$

we can be written as follows:

$$\tilde{u}(x,y,t;r) = [(r-7)^n, (-5-r)^n] \oplus ((x+y)E_\alpha(3t^\alpha)), \quad 0 \le r \le 1.$$

Example 4.3. Consider the following three-dimensional fuzzy fractional diffusion equation. Suppose that $D = 1$, $F(x, y, z) = -(x, y, z)$, $\Psi = [0, 1] \times [0, 1] \times [0, 1]$, then equations (3.1), we obtain

$$\begin{aligned}
\frac{\partial^\alpha \tilde{u}(x,y,z,t)}{\partial t^\alpha} &= \Delta\tilde{u}(x,y,z,t) \oplus x \odot \frac{\partial \tilde{u}(x,y,z,t)}{\partial x} \oplus y \odot \frac{\partial \tilde{u}(x,y,z,t)}{\partial y} \oplus z \odot \frac{\partial \tilde{u}(x,y,z,t)}{\partial z} \oplus 3 \odot \tilde{u}(x,y,z,t),\\
& 0<\alpha\le 1, \quad x\ge 0, \quad y \ge 0, \quad z \ge 0.
\end{aligned} \tag{4.19}$$

with the initial conditions

$$\tilde{u}(x,y,z,0) = [(2r-2)^n, (2-2r)^n] \odot (x+y+z)^2, \text{where} \quad x,y,z\in[0,1], \quad (n=1,2,3,\ldots), \tag{4.20}$$

and the boundary conditions

$$\begin{cases}
\tilde{u}(0,y,z,t) = (3+(y+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha), & t\ge 0,\\
\tilde{u}(1,y,z,t) = (3+(1+y+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha), & t\ge 0,\\
\tilde{u}(x,0,z,t) = (3+(x+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha), & t\ge 0,\\
\tilde{u}(x,1,z,t) = (3+(x+1+z)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha), & t\ge 0,\\
\tilde{u}(x,y,0,t) = (3+(x+y)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha), & t\ge 0,\\
\tilde{u}(x,y,1,t) = (3+(x+y+1)^2)E_\alpha(5t^\alpha) - 3E_\alpha(3t^\alpha), & t\ge 0,
\end{cases} \tag{4.21}$$

The parametric form of (4.19) is

$$\begin{aligned}
\frac{\partial^\alpha \underline{u}(x,y,z,t)}{\partial t^\alpha} &= \Delta\underline{u}(x,y,z,t) + x\frac{\partial \underline{u}(x,y,z,t)}{\partial x} + y\frac{\partial \underline{u}(x,y,z,t)}{\partial y} + z\frac{\partial \underline{u}(x,y,z,t)}{\partial z} + 3\underline{u}(x,y,z,t),\\
& 0<\alpha\le 1,
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\frac{\partial^\alpha \overline{u}(x,y,z,t)}{\partial t^\alpha} &= \Delta\overline{u}(x,y,z,t) + x\frac{\partial \overline{u}(x,y,z,t)}{\partial x} + y\frac{\partial \overline{u}(x,y,z,t)}{\partial y} + z\frac{\partial \overline{u}(x,y,z,t)}{\partial z} + 3\overline{u}(x,y,z,t),\\
& 0<\alpha\le 1,
\end{aligned} \tag{4.23}$$

for $r \in [0, 1]$, where $\underline{u}$ stands for $\underline{u}(x, y, z, t; r)$, and similar to $\overline{u}$.
where

$$E_\alpha(t) = \sum_{j=0}^{\infty}\frac{t^j}{\Gamma(1+j\alpha)}, \quad \alpha > 0,$$

is the Mittag-Leffer function in one parameter.

Substituting the iteration formulation (3.11) and (3.12) with the initial condition (4.20), we get

$$\left.\begin{aligned}
&q^0 : \underline{u}_0(x,y,z,t;r) = (2r-2)^n(x+y+z)^2\\
&q^1 : \underline{u}_1(x,y,z,t;r) = \frac{(2r-2)^n(6+5(x+y+z)^2)t^{\alpha}}{\Gamma(1+\alpha)},\\
&q^2 : \underline{u}_2(x,y,z,t;r) = \frac{(2r-2)^n(48+25(x+y+z)^2)t^{2\alpha}}{\Gamma(1+2\alpha)},\\
&q^3 : \underline{u}_3(x,y,z,t;r) = \frac{(2r-2)^n(294+125(x+y+z)^2)t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\qquad\vdots\\
&q^j : \underline{u}_j(x,y,z,t;r) = \frac{(2r-2)^n(3(5^j-3^j)+5^j(x+y+z)^2)t^{j\alpha}}{\Gamma(1+j\alpha)},
\end{aligned}\right\} \tag{4.24}$$

and

$$\left.\begin{aligned}
&q^0 : \overline{u}_0(x,y,z,t;r) = (2-2r)^n(x+y+z)^2\\
&q^1 : \overline{u}_1(x,y,z,t;r) = \frac{(2-2r)^n(6+5(x+y+z)^2)t^{\alpha}}{\Gamma(1+\alpha)},\\
&q^2 : \overline{u}_2(x,y,z,t;r) = \frac{(2-2r)^n(48+25(x+y+z)^2)t^{2\alpha}}{\Gamma(1+2\alpha)},\\
&q^3 : \overline{u}_3(x,y,z,t;r) = \frac{(2-2r)^n(294+125(x+y+z)^2)t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\qquad\vdots\\
&q^j : \overline{u}_j(x,y,z,t;r) = \frac{(2-2r)^n(3(5^j-3^j)+5^j(x+y+z)^2)t^{j\alpha}}{\Gamma(1+j\alpha)},
\end{aligned}\right\} \tag{4.25}$$

Therefore, we obtain a compact form

$$\underline{u}(x,y,z,t;r) = \lim_{n\to\infty}\sum_{j=0}^{n}\underline{u}_j(x,y,z,t;r) = (2r-2)^n\left(\lim_{n\to\infty}\sum_{j=0}^{n}\frac{(3(5^j-3^j)+5^j(x+y+z)^2)t^{j\alpha}}{\Gamma(1+j\alpha)}\right),$$

$$\overline{u}(x,y,z,t;r) = \lim_{n\to\infty}\sum_{j=0}^{n}\overline{u}_j(x,y,z,t;r) = (2-2r)^n\left(\lim_{n\to\infty}\sum_{j=0}^{n}\frac{(3(5^j-3^j)+5^j(x+y+z)^2)t^{j\alpha}}{\Gamma(1+j\alpha)}\right),$$

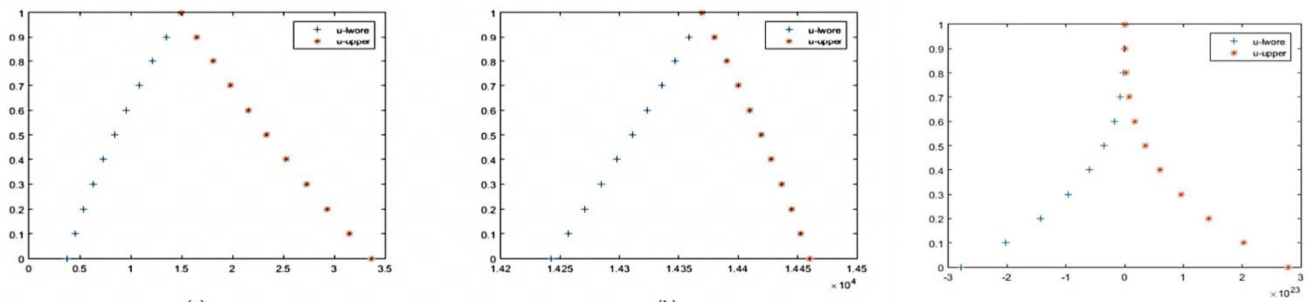
we can be written as follows:

$$\tilde{u}(x,y,z,t;r) = [(2r-2)^n,(2-2r)^n]\odot\left((3+(x+y+z)^2)E_{\alpha}(5t^{\alpha})-3E_{\alpha}(3t^{\alpha})\right),\quad 0\le r\le 1.$$

Figure (1) is used to illustrate that, the left-hand functions of the *r*-level set of $\tilde{u}$ (u lower) are always increasing functions of *r* and the right-hand functions of the *r*-level set of $\tilde{u}$ (u upper) are always decreasing functions of *r* in the above-mentioned examples

V. CONCLUSION

In this article, the fuzzy fractional variational homotopy perturbation iteration method (FFVHPIM) has been successfully applied for solving the fuzzy fractional diffusion equation. The new method is investigated based on the strongly gH-differentiable and Riemann-Liouville gH-differentiability. Using this method, a fast affinity sequence can be obtained that tends to accurately solve the equation. The results show that FFVHPIM is a very effective, convenient and accurate mathematical tool for solving the fuzzy fractional diffusion equation. Finally, three examples are used to illustrate the main results of this article.



(a) (b) (c)

Figure 1: Ex (1): t = 0.001, x = 0.4, n = 2, a = 1/2, Ex (2): t = 3, x = 0.2, y = 0.7 n = 3, a = 1/2. Ex (3): t = 5, x = 0.1, y = 0.2, z = 0.3, n = 3, a = 1/2.

VI. REFERENCES

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